

On linear differential equations with reductive Galois group

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Abstract

Given a connection on a meromorphic vector bundle over a compact Riemann surface with reductive Galois group, we associate to it a projective variety. Connections such that their associated projective variety are curves can be classified, up to projective equivalence, using ruled surfaces. In particular, such a meromorphic connection is the pullbacks of a Standard connection. This extend a similar result by Klein for second-order ordinary linear differential equations to a broader class of equations.

Introduction

In [1], [2] Baldassari and Dwork give a contemporary formulation of a result known to Klein [9], [10] on second order ordinary linear differential equations with algebraic solutions. The result is most easily stated in terms of projective equivalence (cf. Definition 6).

Theorem 1. *If an ordinary second order linear differential equation with rational coefficients has finite projective Galois group, it is projectively equivalent to a pullback, by a rational map, of a hypergeometric equation.*

The collection of hypergeometric equations appearing in Klein's theorem can be classified using Schwartz triples as they correspond to Galois coverings of the Riemann Sphere by another Riemann Sphere. For each finite group in $PGL_2(\mathbb{C})$ there is one hypergeometric equation.

In broad terms the argument is as follows: let y_1 and y_2 be two \mathbb{C} -linearly independent solutions for an equation satisfying the hypothesis of the theorem and let $G \subseteq GL_2(\mathbb{C})$ be its differential Galois group. The covering and the pullback from the theorem arise by taking the composition

$$t : \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})/G \simeq \mathbb{P}^1(\mathbb{C})$$

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given by:

$$\begin{array}{ccc}
 & (y_1(x) : y_2(x)) & \mathbb{P}^1(\mathbb{C}) \\
 & \downarrow & \downarrow \\
 (x : 1) \dashrightarrow \frac{-}{t} \dashrightarrow (y_1 : y_2) \cdot G & & \mathbb{P}^1(\mathbb{C})/G
 \end{array}$$

where G is acting by Möebius transformations on $\mathbb{P}^1(\mathbb{C})$. If t is an isomorphism then the equation is hypergeometric; if not, then it gives the rational pullback map. When t is an isomorphism, the hypergeometric functions $(y_1 : y_2)$ locally give sections of the covering $\mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})/G$.

An extension of this result for the third order case was obtained by M. Berkenbosch [3], where he also gives an algorithmic implementation of Klein's result simplifying Kovacic's algorithm [11]. Berkenbosch introduces the concept of "standard equations" in order to state his generalization. A standard equation is an ordinary linear differential equation which is minimal in the sense that any other ordinary linear differential equation with finite projective Galois group must be a pullback thereof. The collection of standard equations is infinite (even for a fixed group G !) and so far it lacked of structure; therefore classifying these equations has not been done yet.

The purpose of this article is to formulate an extension of Klein's theorem for equations of arbitrary order that also covers many non-algebraic cases and to give a classification of the standard equations using ruled surfaces. We treat the problem in terms of differential modules and connections.

Our main tool in achieving this extension is Compoint's theorem. This result gives a very concrete description of the maximal differential ideal involved in the construction of a Picard-Vessiot extension for the connection. In the first part of this article we introduce the geometric concepts involved. In the second we give an algebraic interpretation of these concepts. In the third and final part, we prove the generalization and introduce the classifying ruled surfaces. In the appendix we study some of the consequences for the algebraic case.

We finish this introduction by studying and re-interpreting the original mapping considered by Klein [1, 2, 9, 10] and Fano [6] so that our classifying ruled surface becomes apparent.

Motivation: The algebraic case

Consider an irreducible homogeneous ordinary linear differential equation

$$L(y) = \frac{d^n}{dz^n}y + a_{n-1}\frac{d^{n-1}}{dz^{n-1}}y + \dots + a_1\frac{d}{dz}y + a_0y = 0,$$

with $a_{n-1}, \dots, a_1, a_0 \in \mathbb{C}(z)$. Let $U \subseteq \mathbb{C}$ be an open set avoiding the set of singularities S of $L(y) = 0$ and admitting n \mathbb{C} -linearly independent solutions

$$y_i : U \longrightarrow \mathbb{C}, \quad i \in \{0, 1, \dots, n-1\}.$$

These n solutions define an analytic map

$$\begin{aligned} U &\longrightarrow \mathbb{P}^{n-1}(\mathbb{C}) \\ z &\longmapsto (y_0(z) : y_1(z) : \dots : y_{n-1}(z)). \end{aligned}$$

By analytic continuation, we can extend this map to a multi-valued map on all $\mathbb{C} \setminus S$. The map is multi-valued up to monodromy, therefore, as the differential Galois group $G \subseteq GL_n(\mathbb{C})$ of the equation contains its monodromy group, our analytic continuation becomes single-valued if we post-compose it by the quotient map

$$\mathbb{P}^{n-1}(\mathbb{C}) \longrightarrow \mathbb{P}^{n-1}(\mathbb{C})/G,$$

where the linear group G is acting by projective linear automorphisms. We denote the resulting composition by $t : \mathbb{C} \setminus S \rightarrow \mathbb{P}^{n-1}(\mathbb{C})/G$.

Let us assume for the rest of this introduction that G is finite and so it coincides with the monodromy group of $L(y) = 0$. In particular, under such assumption, the y_i 's are algebraic over $\mathbb{C}(z)$.

As we pointed out, in the case where $n = 2$, t is injective implies that $L(y) = 0$ is a hypergeometric equation. Berkenbosch's idea was to consider the case when $n = 3$ calling $L(y) = 0$ a standard equation if t is injective [3]. So with Berkenbosch's approach the standard equation would no longer be identified with the quotient map $\mathbb{P}^2(\mathbb{C}) \rightarrow \mathbb{P}^2(\mathbb{C})/G$, but by the pair: the quotient map together with the Zariski closure of the image of the multi-valued analytic map

$$\begin{aligned} \mathbb{C} \setminus S &\longrightarrow \mathbb{P}^2(\mathbb{C}) \\ z &\longmapsto (y_0(z) : y_1(z) : y_2(z)). \end{aligned}$$

To obtain the quotient map one just needs to know the Galois group of $L(y) = 0$, to obtain the genus of the closure of the image one relies on the exponents of the singularities S (see Appendix). The drawback of this approach is that although the genus of the image does not depend on the choice of the y_i 's, the homogeneous polynomial $P[X_0, X_1, X_2]$ vanishing at the triple (y_0, y_1, y_2) does, making hard to identify exactly which standard equation it corresponds to, complicating the identification of the map t and, therefore, any algorithmic implementation of the result.

In this paper we will deviate from this approach and work instead with line bundles and ruled surfaces. First, note that since t is algebraic then we can extend its domain from $\mathbb{C} \setminus S$ to $\mathbb{P}^1(\mathbb{C})$. Secondly, the map t defines a line bundle \mathcal{L} (cf. [12, Section II.7]). Thirdly, we can obtain a second line bundle \mathcal{L}' by working with the map t_1 defined similarly to t but using the n -tuple $(y'_0, y'_1, \dots, y'_{n-1})$ instead of $(y_0, y_1, \dots, y_{n-1})$. Finally, the projective space bundle defined by the rank-2 vector bundle $\mathcal{L} \oplus \mathcal{L}'$ will characterize the standard equation. Those interested in seeing which ruled surfaces correspond to the second order equations can jump to the Appendix and then comeback to read the rest of the paper.

To deal with the non-algebraic cases we will rely on matrix differential equations instead of linear differential equations. So instead of using the map given by n linearly independent solutions, we will use the map given by the full system of solutions. In order to obtain a coordinate-free description we will rely on the concept of connections.

1 Geometric considerations

We first establish notation. X denotes a compact Riemann surface, $k := \mathbb{C}(X)$ is the associated field of meromorphic functions, and we consider a rank n meromorphic vector bundle over X induced by a holomorphic vector bundle $\Pi : E \rightarrow X$ (cf. [14]). By abuse of notation we will use $\Pi : E \rightarrow X$ to describe both: the holomorphic and the induced meromorphic vector bundle.

The sheaf of meromorphic sections of Π will be denoted by \mathcal{E} , and the sheaf of meromorphic functions over X by \mathcal{M} . The concept of the sheaf \mathcal{E} of meromorphic sections of a holomorphic vector bundle can be found in full detail in [7]. The sheaf of meromorphic 1-forms, and the sheaf of meromorphic tangent fields, will be denoted by $\Omega_{\mathcal{M}}^1$ and $\mathcal{T}X$ respectively. The sheaf of differential forms $\Omega_{\mathcal{M}}^1$ is the meromorphic dual of $\mathcal{T}X$ (cf. [14]). Given an $f \in k$ there is a global meromorphic differential form $df \in \Omega_{\mathcal{M}}^1(X)$ defined as follows:

$$\begin{aligned} df : \mathcal{T}X(X) &\longrightarrow k \\ v &\longmapsto df(v) : p \mapsto v_p(f). \end{aligned}$$

Any global tangent field $v \in \mathcal{T}X(X)$ induces a derivation in k , i.e. the map

$$\begin{aligned} v : k &\longrightarrow k \\ f &\longmapsto v(f) : p \mapsto v_p(f) \end{aligned}$$

is additive and satisfies the Leibniz rule:

$$\begin{aligned} v(f + g) &= v(f) + v(g) \\ v(fg) &= v(f)g + fv(g), \quad \forall (f, g) \in k^2. \end{aligned}$$

Once we fix v , the field k together with the derivation defined by v is a differential field. The field of complex numbers \mathbb{C} can be identified with a subfield of k by regarding the complex numbers as constant functions. With this identification in mind we see that the kernel of v , known as the *constants* of the differential field, is \mathbb{C} provided $v \neq 0$. For let $x \in k$ be such that x is not a constant, then k is an algebraic extension of $\mathbb{C}(x)$. Furthermore the derivation d/dx of $\mathbb{C}(x)$ extends uniquely to a no-new-constants derivation v_x of k [16, Exercises 1.5.3], that is $\{f \in k \mid v_x(f) = 0\} = \mathbb{C}$. Thus if $v \neq 0$, since $\mathcal{T}X(X) \simeq k$, there exists a unique $h \in k^*$ such that $v = hv_x$. So the constants of v and the constants of v_x coincide.

Remark 1. In broad terms what we do in this article is to study the following geometric construction. Consider a matrix differential equation

$$v(f^i) = a_j^i f^j, \quad i \in \{1, \dots, n\}, \quad a_j^i \in k$$

and an open $U \subset X$ over which we have a full-system of solutions (y_j^i) , i.e. $y_j^i \in \mathcal{M}(U)$ for $i, j \in \{1, \dots, n\}$ and $\det(y_j^i) \neq 0$. The analytic map

$$\begin{aligned} U &\longrightarrow GL_n(\mathbb{C}) \\ p &\longmapsto (y_j^i(p)) \end{aligned}$$

induces an algebraic map

$$\begin{aligned} U &\longrightarrow GL_n(\mathbb{C})/G \\ p &\longmapsto (y_j^i(p)) \cdot G \end{aligned}$$

where G is the Galois group of our linear differential equation which we will assume reductive.

Indeed, $\mathbb{C}[GL_n/G] \simeq \mathbb{C}[X_j^i]^G$ (Hilbert 14th), so let P_1, \dots, P_r be a set of generators of $\mathbb{C}[X_j^i]^G$. In the coordinate system (P_1, \dots, P_r) of $GL_n(\mathbb{C})/G$, the map $p \mapsto (y_j^i(p)) \cdot G$ is given by $p \mapsto (P_1(y_j^i(p)), \dots, P_r(y_j^i(p)))$; and by Galois Correspondence $P_l(y_j^i) = f_l \in k$. Therefore in the this coordinate system, $p \mapsto (y_j^i(p)) \cdot G$ is given by $p \mapsto (f_1(p), \dots, f_r(p))$.

Because the last map is algebraic, it can be extended to a meromorphic map defined globally over X . The idea is to see to which extend this last map characterizes our differential equation.

Remark 2. The maps above, $p \mapsto (y_j^i(p))$ and $p \mapsto (y_j^i(p)) \cdot G$, depend on the choices of a full-system of solutions and of an open set U . Therefore, the first thing we will do is to argue bi-rationally that the geometric properties of the image of this maps does not depend on this choices (see Remark 17). For that we will need to provide a coordinate-free description of this image, which will be done by taking a symmetric algebra characterizing the image of the map (Definition 4 and Definition 5).

1.1 Differential Modules

Remark 3. We will use Einstein's notation for indices.

Fix a non-trivial derivation $v \in \mathcal{T}X(X)$ of k . We recall briefly the concept of differential module, which is a coordinate-free description of a matrix differential equation. A more detailed exposition may be found in [16].

Definition 1. A differential k -module (rigorously a (k, v) -module) is a finite dimensional k -vector space M together with an additive map

$$\partial : M \rightarrow M$$

satisfying the Leibnitz rule:

$$\begin{aligned}\partial(m_1 + m_2) &= \partial m_1 + \partial m_2 & \forall (m_1, m_2) \in M^2 \\ \partial fm &= v(f)m + f\partial m & \forall (f, m) \in k \times M\end{aligned}$$

An $m \in M$ such that $\partial m = 0$ is called a horizontal element.

Remark 4. If f is a constant, i.e. if $v(f) = 0$, the Leibnitz rule implies that $\partial fm = f\partial m$. The collection of horizontal elements thus forms a vector space over the field of constants \mathbb{C} .

Remark 5. Fix a basis e_1, \dots, e_n of M and set

$$\partial e_j = -a_j^i e_i \quad \forall j \in \{1, \dots, n\}.$$

If $m = f^i e_i$, then

$$\begin{aligned}\partial m &= v(f^i) e_i + f^i \partial e_i \\ &= v(f^i) e_i - f^i a_i^j e_j \\ &= (v(f^i) - a_j^i f^j) e_i.\end{aligned}$$

Solving the equation $\partial m = 0$ therefore amounts to solve the matrix differential equation

$$v(f^i) = a_j^i f^j, \quad i \in \{1, \dots, n\}.$$

Proposition 2. Let (M, ∂) , (M_1, ∂_1) and (M_2, ∂_2) be three differential modules, then:

i) The tensor product $M_1 \otimes M_2$ inherits a differential k -modules structure under the map:

$$\partial_1 \otimes \partial_2 : m_1 \otimes m_2 \longrightarrow \partial_1 m_1 \otimes m_2 + m_1 \otimes \partial_2 m_2$$

ii) The symmetric power $\text{Sym}^d(M)$ inherits a differential k -module structure as a quotient of the tensor product of d copies of M , $M^{\otimes d}$.

iii) The exterior power $\bigwedge^d M$ inherits a differential k -module structure as a quotient of $M^{\otimes d}$.

iv) The dual $M^* = \text{Hom}_k(M, k)$ inherits a differential k -module structure under the map:

$$\partial^* : \mu \longmapsto [m \mapsto v(\mu(m)) - \mu(\partial m)]$$

v) The space of k -linear morphisms $\text{Hom}_k(M_1, M_2) = M_1^* \otimes M_2$ inherits a differential k -module structure $\partial_1^* \otimes \partial_2$. In particular if $H \in \text{Hom}_k(M_1, M_2)$ is such that $(\partial_1^* \otimes \partial_2)H = 0$ then $\partial_2 \circ H = H \circ \partial_1$.

Proof. Items *i*), *ii*), *iii*), *iv*) are in [16]. To see *v*), write $H = f^i \otimes e_i$ for some $f^i \in M_1^*$ and some $e_i \in M_2$, so that if $m \in M_1$ then $H(m) = f^i(m)e_i$. Now by hypothesis $\partial_1^* f^i \otimes e_i + f^i \otimes \partial_2 e_i = 0$, therefore

$$\begin{aligned} H(\partial_1 m) &= f^i(\partial_1 m)e_i \\ &= v(f^i(m))e_i - \partial_1^* f^i(m)e_i \\ &= v(f^i(m))e_i + f^i(m)\partial_2 e_i \\ &= \partial_2 H(m) \end{aligned}$$

□

Remark 6. *Summarizing, a differential k -module endows a canonical differential structure on any tensorial construction (duals, tensor products, symmetric powers, exterior powers, sums, ...) over k .*

Remark 7. *Given a differential k -module M , we will not always be able to find a basis composed of horizontal elements.*

The definition of a differential k -module depends on our choice of a derivation on k (cf. Definition 1), in our case it was v . Since we are in the realm of Riemann surfaces, which are actually one dimensional manifolds over the complex numbers, we can circumvent this restriction using connections.

1.2 Connections and Pullbacks

A meromorphic connection is a \mathbb{C} -linear map (linear over the constants)

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{M}} \Omega_{\mathcal{M}}^1$$

satisfying the Leibnitz rule

$$\nabla(fV) = V \otimes df + f\nabla V \quad \forall (f, V) \in k \times \mathcal{E}(X)$$

(recall \mathcal{E} is the sheaf of meromorphic sections of the vector bundle $\Pi : E \rightarrow X$, \mathcal{M} the sheaf of meromorphic functions and $\Omega_{\mathcal{M}}^1$ the sheaf of meromorphic differential forms).

Given a meromorphic tangent field v , we define the \mathcal{M} -linear contraction map by

$$\begin{aligned} \iota_v : \mathcal{E} \otimes_{\mathcal{M}} \Omega_{\mathcal{M}}^1 &\longrightarrow \mathcal{E} \\ V \otimes \omega &\longmapsto \omega(v)V. \end{aligned}$$

Not every element in $\mathcal{E} \otimes_k \Omega_{\mathcal{M}}^1$ can be written in the form $V \otimes \omega$, but as the map is \mathcal{M} -linear, it suffices to define the map in such elements. The composition of ∇ followed by this contraction map is denoted by ∇_v . This is commonly called the *covariant derivative* along v .

Remark 8. The k -vector space of global sections $\mathcal{E}(X)$ of Π is isomorphic to k^n , so $(\mathcal{E}(X), \nabla_v)$ is a differential k -module.

Proposition 3. The connection ∇ induces a (k, v) -module structure on $\mathcal{E}(X)$ under the map ∇_v . Conversely a (k, v) -module structure on $\mathcal{E}(X)$ determines a connection on \mathcal{E} .

Proof. Derivations on k are in a natural bijective correspondence with global sections of $\mathcal{T}X$, and $\mathcal{T}X(X) \simeq k$. Suppose $\nu \in \Omega_{\mathcal{M}}^1$ is the dual of a non-zero derivation $v \in \mathcal{T}X$, i.e. $\nu(v) = 1$. Because $\Omega_{\mathcal{M}}^1$ is one-dimensional, for every $\omega \in \Omega_{\mathcal{M}}^1$ we have $\omega = \omega(v)\nu$, and we conclude that tensoring with ν is the inverse to ι_v . Define the following map ∇' :

$$\begin{aligned} \nabla' : \mathcal{E} &\longrightarrow \mathcal{E} \underset{\mathcal{M}}{\otimes} \Omega_{\mathcal{M}}^1 \\ V &\longmapsto \nabla_v V \otimes \nu. \end{aligned}$$

Then

$$\begin{aligned} \nabla' V &= \nabla_v V \otimes \nu \\ &= \iota_v(\nabla V) \otimes \nu \\ &= \nabla V. \end{aligned}$$

Endowing $\mathcal{E}(X)$ with a differential k -module structure is thereby seen to be the same as defining a meromorphic connection over \mathcal{E} . \square

Remark 9. As a corollary of the previous proposition we obtain that every tensorial construction over meromorphic vector bundles with connections inherits a connection in a similar way as it happens with differential modules. This last remark is actually independent of the fact that X is one-dimensional.

We now define connection pullbacks.

Definition 2. Take a meromorphic vector bundle $\Pi_0 : E_0 \rightarrow X_0$ over a compact Riemann surface together with a meromorphic connection ∇_0 and a morphism $f : X \rightarrow X_0$. If $\Pi : E \rightarrow X$ is the pullback bundle of $\Pi_0 : E_0 \rightarrow X_0$, and $(\bar{f}, f) : (E, X) \rightarrow (E_0, X_0)$ stands for the canonical vector bundle morphism (i.e. $\Pi_0 \circ \bar{f} = f \circ \Pi$):

$$\begin{array}{ccc} & E & \\ \bar{f} \swarrow & & \downarrow \\ E_0 & & X \\ \downarrow & \searrow f & \\ X_0 & & \end{array}$$

the pullback (connection) $f^*\nabla_0$ at $p \in X$ is given by

$$[(f^*\nabla_0)_v V](p) = [(\nabla_0)_{f_*v} \bar{f}^* V](f(p))$$

(the vector field f_*v is well defined only locally, but since $(\nabla_0)_\bullet$ is tensorial on Π , the value of $[(\nabla_0)_{f_*v}\bar{f}V](f(p))$ is uniquely determined by $f_*v(p)$).

Remark 10. In terms of sections of vector bundles, it follows from the definition that ∇ is equal to $f^*\nabla_0$ if and only if the section V of Π is horizontal (i.e. $\nabla V = 0$) is equivalent to $\bar{f}V$, as a section of Π_0 , is horizontal.

1.3 Symmetric algebra of first integrals and Fano curve

We now study symmetric products and duals of vector bundles with connections in more detail. For the remainder of this section we set $M = \mathcal{E}(X)$ and we fix a non-zero derivation $v \in \mathcal{T}X(X)$.

Definition 3. Let $\phi \in M^* = \text{Hom}_k(\mathcal{E}(X), k)$. We say that ϕ is a linear first integral (of (E, ∇)) if $\phi(V) \in k$ is constant whenever V is horizontal.

Remark 11. Let ϕ be a linear first integral and let V be horizontal. Then

$$\begin{aligned} 0 &= v(\phi(V)) \\ &= [\nabla_v^* \phi](V). \end{aligned}$$

Therefore, by taking a full system of solutions we see that $\nabla_v^* \phi = 0$ if and only if ϕ is a linear first integral. In particular, linear first integrals form a vector space over the constants.

Definition 4. Denote by $S_k^d(M)$ the \mathbb{C} -vector space of linear first integrals of the d -th symmetric product of (E, ∇) . As a convention we set $S_k^0(M) = \mathbb{C}$. We define the graded \mathbb{C} -algebra of linear first integrals of M as

$$S_k(M) = \bigoplus_{d \geq 0} S_k^d(M).$$

Remark 12. If $V \in M$ is horizontal, then so is V^d in the d -th symmetric power. Thus, given $\phi \in S_k^d(M)$, $\phi(V^d)$ is constant. The elements in $S_k^d(M)$ are called (d -th order) first integrals of M . Once we fix a basis of M^* , i.e. a coordinate system for M , an element in $S_k^d(M)$ is given by a homogeneous polynomial of order d in the coordinates on M . With this in mind $S_k(M)$ corresponds to the collection of rational \mathbb{C} -valued functions over E which contain horizontal sections of \mathcal{E} within their level sets.

Remark 13. If we pick a $V \in M$ we obtain a homomorphism

$$V : S_k(M) \longrightarrow k$$

of \mathbb{C} -algebras by evaluating each first integral in $S_k^d(M)$ at V^d .

Remark 14. Let $U \subseteq X$ be an open set and pick $V \in \mathcal{E}(U)$, then we can evaluate first integrals in $S_k^d(M)$ at V to obtain an element of $\mathcal{M}(U)$. Indeed, since the first integrals in $S_k^d(M)$ are globally defined meromorphic functions,

we can restrict them to $\Pi^{-1}U$ and evaluate them at V^d . So in this case V can be identified with a homomorphism

$$V : S_k(M) \longrightarrow \mathcal{M}(U)$$

of \mathbb{C} -algebras. Note that the previous remark is the particular case $U = X$.

Remark 15. The fundamental theorem of ordinary differential equations guarantees that if p is not a singular point of ∇ , then for a sufficiently small open set $U \subseteq X$ containing p we can find a frame (V_1, \dots, V_n) of $\mathcal{E}(U)$ composed of horizontal elements.

Definition 5. Let H be an invertible element of the differential k -module

$$\text{Hom}_k(\mathcal{E}(U), \mathcal{E}(U)) \simeq [\mathcal{E}^* \otimes_{\mathcal{M}} \mathcal{E}](U)$$

associating to a global frame of $M = \mathcal{E}(X)$, restricted to U , a frame of $\mathcal{E}(U)$ composed of horizontal sections (fixing a basis for M^* , H corresponds to the matrix of coordinates of a full system of solutions). The Fano curve of (E, ∇) is defined as the $\mathcal{M}(U)$ -valued point

$$H : S_k(\text{Hom}_k(\mathcal{E}(X), \mathcal{E}(X))) \longrightarrow \mathcal{M}(U)$$

(see Remark 14). If $S_k(\text{Hom}_k(\mathcal{E}(X), \mathcal{E}(X)))$ is finitely generated we define the projective Fano curve X_0 to be the non-singular model of the projective variety defined by the maximal homogeneous ideal contained in the kernel of the Fano curve (i.e. the maximal homogeneous ideal contained in the kernel of H).

Remark 16. To obtain the polynomials defining the projective Fano curve we rely on the algorithm by M. van Hoeij and J.-A. Weil [13]. Using their terminology these polynomials correspond to the homogeneous “invariants” with vanishing “dual first integrals”.

Remark 17. There are many aspects of this definition that require elaboration.

- The Fano curve is k -valued: this is a consequence of the Galois correspondence. This fact, together with the ideal defining the Fano curve will be studied in the next section (Proposition 6).
- The Fano curve is independent of the choice of H up to isomorphism: if \tilde{H} is another invertible element of $\text{Hom}_k(\mathcal{E}(U), \mathcal{E}(U))$ associating to a global frame of $M = \mathcal{E}(X)$ a frame of $\mathcal{E}(U)$ composed of horizontal sections, then it differs from H by an element of $GL_n(\mathbb{C})$ multiplying on the right and an element of $GL_n(k)$ multiplying on the left, which one can see them as acting on $S_k(\text{Hom}_k(\mathcal{E}(U), \mathcal{E}(U)))$ sending one Fano curve to another.
- The Fano curve is also independent of the choice of U up to isomorphism: let \tilde{U} be another open set where one can define a frame composed of horizontal sections, then by taking a path from U to \tilde{U} we can prolong the frame holomorphically to a frame over \tilde{U} . Because the first integrals are constant over horizontal sections and are globally defined, the prolongation does not change the k -valued point H (cf. first • in this Remark).

Remark 18. *Let us see how we can obtain geometrically the projective Fano curve under the assumption that the Galois group G of the equation is reductive. Fixing a basis for M let H be represented over the open set $U \subseteq X$ by the full-system of solutions (y_j^i) , i.e. $y_j^i \in \mathcal{M}(U)$ for $i, j \in \{1, \dots, n\}$. The image of the analytic map*

$$\begin{aligned} U &\longrightarrow GL_n(\mathbb{C}) \\ p &\longmapsto (y_j^i(p)) \end{aligned}$$

corresponds to a solution curve in the phase portrait. Composing this map with the canonical projection $GL_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C})/G$, we obtain an algebraic map

$$\begin{aligned} U &\longrightarrow GL_n(\mathbb{C})/G \\ p &\longmapsto (y_j^i(p)) \cdot G \end{aligned}$$

Therefore the Zariski closure of the image is an algebraic curve. The projective Fano curve corresponds to the projective variety defined by the cone over this algebraic curve. We will establish the invariance of the Fano curve under “projective equivalence” (Proposition 10).

1.4 Projective equivalence

Assume now that $P : L \rightarrow X$ is a 1-dimensional vector bundle (a line bundle), and fix a global non-zero meromorphic section $s \in \mathcal{L}(X)$. We can then identify \mathcal{E} with $\mathcal{L} \otimes \mathcal{E}$ through the morphism $V \mapsto s \otimes V$. It must be noted that this identification is not unique, since it depends on the choice of s .

Definition 6. *Given another meromorphic connection ∇' on $\Pi : E \rightarrow X$, we say that ∇ and ∇' are projectively equivalent if there exist a 1-dimensional meromorphic vector bundle $P : L \rightarrow X$ with a connection ∇_1 such that*

$$\nabla' \simeq \nabla_1 \otimes \nabla.$$

Remark 19. *Assume that $s \otimes \cdot : V \mapsto s \otimes V$ is a horizontal morphism from (\mathcal{E}, ∇') to $(\mathcal{L} \otimes \mathcal{E}, \nabla_1 \otimes \nabla)$, then*

$$s \otimes \nabla'(V) = \nabla_1 s \otimes V + s \otimes \nabla V.$$

Proposition 4. *Projective equivalence is an equivalence relation on the collection of connections over \mathcal{E} .*

Proof. Reflexivity and Transitivity is immediate. We prove symmetry. Note that since $(P, \nabla_1) : L \rightarrow X$ is 1-dimensional, the same holds for the dual (L^*, P^*, ∇_1^*) . Under the canonical isomorphisms we have

$$\mathcal{L}^* \otimes \mathcal{L} \simeq \text{Hom}_{\mathcal{M}}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{M}.$$

Moreover in terms of the basis $s = s_1$ of $\mathcal{L}(X)$, if $s^1 \in \mathcal{L}^*$ is such that $s^1(s_1) = 1$, then

$$\begin{aligned} \{[\nabla_1^* \otimes \nabla_1]_v(s^1 \otimes s_1)\}(s_1) &= \{\nabla_{1v}^* s^1 \otimes s_1 + s^1 \otimes \nabla_{1v} s_1\}(s_1) \\ &= \{\nabla_{1v}^* s^1\}(s_1)s_1 + s^1(s_1)\nabla_{1v} s_1 \\ &= \{v(s^1(s_1)) - s^1(\nabla_{1v} s_1)\}s_1 + \nabla_{1v} s_1 \\ &= \nabla_{1v} s_1 - s^1(\nabla_{1v} s_1)s_1 = 0. \end{aligned}$$

So the connection on $\mathcal{L}^* \otimes \mathcal{L}$ is trivial, and if $\nabla' \simeq \nabla_1 \otimes \nabla$ then

$$\nabla_1^* \otimes \nabla' \simeq \nabla_1^* \otimes \nabla_1 \otimes \nabla \simeq \nabla.$$

We conclude that projective equivalence is an equivalence relation. \square

1.5 The geometric Galois group

Remark 20. Let $\phi \otimes V \in [\mathcal{E}^* \otimes_{\mathcal{M}} \mathcal{E}](X)$, so that:

$$[\nabla^* \otimes \nabla](\phi \otimes V) = \nabla^* \phi \otimes V + \phi \otimes \nabla V.$$

Then under the canonical isomorphism $\text{Hom}_k(\mathcal{E}(X), \mathcal{E}(X)) \simeq [\mathcal{E}^* \otimes_{\mathcal{M}} \mathcal{E}](X)$ we obtain:

$$\begin{aligned} [\nabla^* \otimes \nabla]_v(\phi \otimes V)(W) &= [\nabla_v^* \phi](W)V + \phi(W)\nabla_v V \\ &= \{v(\phi(W)) - \phi(\nabla_v W)\}V + \phi(W)\nabla_v V \\ &= \nabla_v[\phi(W)V] - \phi(\nabla_v W)V \\ &= \nabla_v[(\phi \otimes V)(W)] - (\phi \otimes V)(\nabla_v W). \end{aligned}$$

This implies that $\psi \in \text{Hom}_k(\mathcal{E}(X), \mathcal{E}(X))$ is horizontal if and only if it is a connection preserving map, i.e.

$$\nabla[\psi(W)] = \psi(\nabla W)$$

(in terms of differential k -modules, H is horizontal if and only if H is a morphism of differential modules).

Fix a horizontal automorphism $\psi \in \text{Hom}_k(\mathcal{E}(X), \mathcal{E}(X))$ (e.g. ψ is the identity). The collection of horizontal automorphisms $\phi \in \text{Hom}_k(\mathcal{E}(X), \mathcal{E}(X))$ defining the same map

$$\phi : \text{S}_k(\text{Hom}_k(\mathcal{E}(X), \mathcal{E}(X))) \longrightarrow \mathbb{C}$$

as ψ will be denoted by $[\psi]$. In particular, the elements in $\psi^{-1}[\psi]$ form a group, which we will call the *geometric Galois group* of ∇ . Note that ϕ is \mathbb{C} -valued because first integrals are constant on horizontal elements.

Remark 21. As in the case of the Fano curve, it follows that the geometric Galois group is independent of ψ and U , up to isomorphism. More generally, we could replace ψ by any (non-necessarily horizontal) automorphism in $\text{Hom}_k(\mathcal{E}(U), \mathcal{E}(U))$. Indeed, in such case we would obtain a conjugate of G in $GL_n(\mathcal{M}(U))$.

Remark 22. *The geometric Galois group measures the horizontal automorphisms that cannot be distinguished one from another by means of first integrals.*

2 Algebraic Interpretation

To give an algebraic interpretation of the geometric objects introduced above, we require a global meromorphic frame $F = (e_1, \dots, e_n) \in \mathcal{E}(X)^n$ of $E \rightarrow X$ so that we can do some computations using coordinates.

As above, we fix a derivation $v \in \mathcal{T}X(X)$. We discussed earlier that the solutions to the equation $\nabla_v V = 0$, are given by the solutions to the matrix differential equation

$$v(f^i) = a_j^i f^j \quad i \in \{1, \dots, n\}, \quad (1)$$

where $f^i = e^i(V)$ and $\nabla_v e_i = -a_i^j e_j$.

2.1 Picard-Vessiot Extensions and Galois groups

Let $H = (V_1, \dots, V_n)$ be a frame of $\mathcal{E}(U)$ composed of horizontal elements. If we denote by $y_j^i = e^i(V_j) \in \mathcal{M}(U)$ the coordinates of these horizontal elements in our original frame F , then

$$v(y_j^i) = a_k^i y_j^k$$

and the Picard-Vessiot extension is given by the subfield

$$K := k(y_j^i)$$

of $\mathcal{M}(U)$. The inclusion map into this extension is given by the restriction map:

$$\begin{aligned} k \simeq \mathcal{M}(X) &\longrightarrow \mathcal{M}(U) \\ f &\longmapsto f|_U. \end{aligned}$$

Formally, the Picard-Vessiot extensions can be obtained as follows (a rigorous exposition may be found in [16]). Consider the ring of polynomials in $n \times n$ variables with coefficients in k ,

$$k[X_j^i, \frac{1}{\det}]_{i,j \in \{1, \dots, n\}},$$

inverting the determinant polynomial $\det := \det(X_j^i)$. We turn this ring into a differential ring extension of (k, v) by setting

$$v(X_j^i) = a_l^i X_j^l$$

and using the Leibniz rule and the quotient rule

$$v(ab^{-1}) = [v(a)b - av(b)]b^{-2}$$

we extend the derivation to the whole ring. An ideal $I \subseteq k[X_j^i, \frac{1}{\det}]$ is differential if it is closed under derivation, i.e. $v(I) \subset I$. Maximal differential ideals are prime. We obtain a Picard-Vessiot extension for ∇ by taking the fraction field of the quotient of $k[X_j^i, \frac{1}{\det}]$ by a maximal differential ideal I .

Note that we can make $\mathrm{GL}_n(\mathbb{C})$ act on $k[X_j^i, \frac{1}{\det}]$ by differential automorphisms over k by setting for $(g_j^i) \in \mathrm{GL}_n(\mathbb{C})$

$$\begin{aligned} (g_j^i) : k[X_j^i, \frac{1}{\det}] &\longrightarrow k[X_j^i, \frac{1}{\det}] \\ X_j^i &\longmapsto X_l^i g_j^l \end{aligned}$$

We can identify the Galois group G with the elements of $\mathrm{GL}_n(\mathbb{C})$ sending I to itself. In particular, we may take I as the kernel of the evaluation map of k -algebras:

$$\begin{aligned} \Psi : k[X_j^i, \frac{1}{\det}] &\longrightarrow K \subseteq \mathcal{M}(U) \\ X_j^i &\longmapsto y_j^i \end{aligned}$$

Let us consider with more care the relationship between the geometric Galois group and the (algebraic) Galois group. As above, H will denote the invertible element $e^i \otimes V_i \in \mathrm{Hom}_k(\mathcal{E}(U), \mathcal{E}(U))$. An element of $S_k^d(\mathrm{Hom}_k(\mathcal{E}(X), \mathcal{E}(X)))$ corresponds on the frame F to a homogeneous polynomial $P(X_j^i)$ of degree d such that $P(y_j^i) = f$, where $f \in k$. So $P(X_j^i) - f$ is in the kernel I of the evaluation map Ψ . Moreover, if $g = (g_j^i) \in \mathrm{GL}_n(\mathbb{C})$ is in our algebraic definition of the Galois group, then $P(X_l^i g_j^l) - f$ is again in I , meaning that $P(y_l^i g_j^l) = f$, so the geometric Galois group contains the (algebraic) Galois group (cf. Remark 21).

Remark 23. In [4, Theorem 4.2] Compoint proves that there is a bijective correspondence between first integrals in $S_k^d(\mathrm{Hom}_k(\mathcal{E}(X), \mathcal{E}(X)))$ and homogeneous elements of $\mathbb{C}[X_j^i]^G$ of degree d . Furthermore, the theorem also states (see Theorem 5 below) that when the (algebraic) Galois group G is reductive and unimodular, the elements of I of the form $P(X_j^i) - f$, where $P(X_j^i) \in \mathbb{C}[X_j^i]^G$ and $P(y_j^i) = f$, generate I , so in such case the geometric and the (algebraic) Galois group coincide. In view of Compoint's result we will restrict ourselves to such case.

2.2 Compoint's Theorem and the projective Fano Curve

In analogy with classical Galois theory, differential Galois theory also admits a Galois correspondence. In particular, the fixed field K^G is the ground field k . Furthermore using the action in the previous section, if $P(X_j^i) \in k[X_j^i]$ is invariant under the action of G , then the Galois correspondence implies $P(y_j^i) \in k$.

Remark 24. For the rest of this section we will assume that G is unimodular and reductive. In particular this will imply that $S_k(\text{Hom}_k(\mathcal{E}(X), \mathcal{E}(X)))$ is finitely generated, and so we will be able to associate to ∇ a projective Fano curve X_0 . We now introduce a theorem that allows us to effectively compute X_0 .

Theorem 5 (Compoint [4]). *If G is reductive and unimodular, then I is generated by the G -invariants it contains. Moreover, if P_0, \dots, P_r is a set of generators for the \mathbb{C} -algebra of G -invariants in $\mathbb{C}[X_j^i]$, and if $f_0, \dots, f_r \in k$ are such that $P_i - f_i \in I$, then I is generated over $k[X_j^i, \frac{1}{\det}]$ by $P_i - f_i$, $i \in \{0, \dots, r\}$.*

Remark 25. Compoint's theorem says that I is uniquely determined by the restriction of $\Psi : k[X_j^i, \frac{1}{\det}] \rightarrow K$ to $\mathbb{C}[X_j^i]^G \rightarrow k$.

$$\begin{array}{ccc} & & K \\ & \nearrow & \downarrow \\ \mathbb{C}[X_j^i] & & k \\ \downarrow & \nearrow & \\ \mathbb{C}[X_j^i]^G & & \end{array}$$

Proposition 6. *We keep the notation and hypotheses of the theorem and the remark. Let J be the maximal homogeneous ideal contained in the kernel of $\Psi \upharpoonright_{\mathbb{C}[X_j^i]^G}$. The projective variety $Z(J) \subseteq \text{Proj}(\mathbb{C}[X_j^i]^G)$ defined by the homogeneous ideal J , is bi-rationally equivalent to the projective Fano curve X_0 of ∇ . Moreover, we have*

$$\mathbb{C}(X_0) = \mathbb{C}\left(\frac{f_i^{m_i}}{f_j^{m_j}} \mid m_i n_i = m_j n_j, f_j \neq 0\right).$$

Proof. Let $P(X_j^i) \in J$ of degree d , then $P(X_j^i)$ corresponds, in our frame F , to a first integral of degree d (see Remark 23), i.e. a homogeneous element $\lambda \in S_k(\text{Hom}_k(\mathcal{E}(X), \mathcal{E}(X)))$, vanishing at $H = (V_1, \dots, V_n) = e^i \otimes V_i$ (being pedantic, it vanishes at H^d). This means that λ is in the homogeneous ideal defining the projective Fano curve. Conversely [4, Theorem 4.2] says that such a λ , i.e. a first integral of degree d vanishing at H , corresponds to a homogeneous polynomial invariant under the G -action of degree d , $P(X_j^i) \in \mathbb{C}[X_j^i]$, such that $P(y_j^i) = 0$, i.e. $P(X_j^i) \in J$.

Fix $P_l \in \{P_0, \dots, P_r\}$ such that $f_l = \psi(P_l) = P_l(y_j^i) \neq 0$, let $U_{P_l} = \{P_l \neq 0\} \subseteq \text{Proj}(\mathbb{C}[X_j^i]^G)$. The ring of coordinate functions of U_{P_l} is

$$\mathbb{C}[U_{P_l}] = \mathbb{C}\left[\frac{P_i^{m_i}}{P_l^{m_l}} \mid m_i n_i = m_l n_l\right],$$

So

$$\mathbb{C}[U_{P_l} \cap X_0] = \mathbb{C}\left[\frac{f_i^{m_i}}{f_l^{m_l}} \mid m_i n_i = m_l n_l\right],$$

The statement follows after taking the quotient field. \square \square

Lemma 7. *We keep the notation and hypotheses of Compoint's theorem. Let $pr : SL_n(\mathbb{C}) \rightarrow PSL_n(\mathbb{C})$ and set*

$$\tilde{G} := pr^{-1}(PG),$$

where $PG = G/\mu_n$ ($\mu_n = Z(SL_n(\mathbb{C}))$). If $P(y_j^i) = 1$ for some homogeneous $P(X_j^i) \in \mathbb{C}[X_j^i]^G$ of degree n , then $\mathbb{C}(X_0) = \mathbb{C}(y_j^i)^{\tilde{G}}$.

Proof. As in the latter proposition we take $U_P = \{P \neq 0\}$, so $X_0 \subseteq U_P$ and $\mathbb{C}[X_0] = \mathbb{C}[f_i^{m_i} | m_i n_i = \text{lcm}(m_i, n)]$. Recall $f_i^{m_i} = P_i^{m_i}(y_j^i)$. Now because $n | \deg(P_i^{m_i})$ then $P_i^{m_i} \in \mathbb{C}[X_j^i]^{\mu_n}$, but $P_i^{m_i} \mathbb{C}[X_j^i]^G$ so $P_i^{m_i} \in \mathbb{C}[X_j^i]^{\tilde{G}}$, whence $\mathbb{C}[X_0] \subseteq \mathbb{C}[y_j^i]^{\tilde{G}}$. Conversely $\mathbb{C}[X_j^i]^{\tilde{G}} \subseteq \mathbb{C}[X_j^i]^G$, so $\mathbb{C}[y_j^i]^{\tilde{G}} \subseteq \mathbb{C}[X_0]$. \square \square

Remark 26. *A particular case of the previous lemma occurs when*

$$\det(y_j^i) = 1.$$

Lemma 8. *Under the hypotheses of Compoint's theorem, if $\det(y_j^i) = 1$ there is a choice of $v \in \mathcal{T}X$ such that in (1) one has*

$$(a_j^i) \in M_{n \times n}(\mathbb{C}(X_0)) + \mathfrak{g}(k),$$

where \mathfrak{g} is the Lie algebra of G .

Proof. Let $P_l = P_l(X_j^i) \in \mathbb{C}[X_j^i]^{\tilde{G}}$, for $l \in \{1, \dots, r\}$ be generators of the \tilde{G} -invariant subalgebra $\mathbb{C}[X_j^i]^{\tilde{G}}$. Let $f_l \in k$ be such that $P_l - f_l \in I$, $l \in \{1, \dots, r\}$. We denote by $\frac{\partial P_l}{\partial X_\kappa^i}(X_j^i)$ the partial derivative of P_l with respect to X_κ^i .

We introduce a differential field $\mathbb{C}(y_j^i)(b_\kappa^i)_{i, \kappa \in \{1, \dots, n\}}$, where the b_κ^i 's are variables, with derivation \tilde{v}_0 defined by:

$$\begin{aligned} \tilde{v}_0(y_j^i) &= b_\kappa^i y_j^k \\ \tilde{v}_0(b_j^i) &= 0. \end{aligned}$$

Using the chain rule we obtain

$$\begin{aligned} \tilde{v}_0(P_l(y_j^i)) &= \frac{\partial P_l}{\partial X_\kappa^i}(y_j^i) \tilde{v}_0(y_\kappa^i) \\ &= \frac{\partial P_l}{\partial X_\kappa^i}(y_j^i) b_\lambda^i y_\kappa^\lambda. \end{aligned}$$

We extend the action of \tilde{G} on $\mathbb{C}(y_j^i)$ to $\mathbb{C}(y_j^i)(b_\kappa^i)_{i, \kappa \in \{1, \dots, n\}}$ by letting each b_κ^i be fixed by \tilde{G} . The chain rule then implies that for $(g_j^i) \in \tilde{G}$ one has

$$\begin{aligned} \tilde{v}_0(P_l(y_i^i g_j^l)) &= \frac{\partial P_l}{\partial X_\kappa^i}(y_i^i g_j^l) \tilde{v}_0(y_\mu^i g_\kappa^\mu) \\ &= \frac{\partial P_l}{\partial X_\kappa^i}(y_i^i g_j^l) b_\lambda^i y_\mu^\lambda g_\kappa^\mu. \end{aligned}$$

The equality $P_l(y_j^i) = P_l(y_l^i g_j^l)$ in turn implies

$$\frac{\partial P_l}{\partial X_\kappa^\iota}(y_j^i) y_\kappa^\lambda = \frac{\partial P_l}{\partial X_\kappa^\iota}(y_l^i g_j^l) y_\mu^\lambda g_\kappa^\mu \quad \forall \iota, \lambda,$$

for all $(g_j^i) \in \tilde{G}$, whence $\frac{\partial P_l}{\partial X_\kappa^\iota}(y_j^i) y_\kappa^\lambda \in \mathbb{C}(y_j^i)^{\tilde{G}}$ for each ι, λ .

Let v be a non-trivial derivation of $\mathbb{C}(X_0) = \mathbb{C}(y_j^i)^{\tilde{G}}$ (unless $\mathbb{C}(X_0) = \mathbb{C}$ in which case we let v be any element in $\mathcal{T}X$). Note that since P_l is \tilde{G} -invariant we have $P_l(y_j^i) = f_l \in \mathbb{C}(X_0) \subseteq \mathbb{C}(y_j^i)$. Consider the following system of linear equations in the variables b_λ^i with coefficients in $\mathbb{C}(X_0)$:

$$\frac{\partial P_l}{\partial X_\kappa^\iota}(y_j^i) y_\kappa^\lambda b_\lambda^i = v(f_l) \quad l \in \{0, \dots, r\}. \quad (2)$$

This system has solutions in k (apply v on both sides of the equalities $P_l(y_j^i) = f_l$ in K). The system of equations is therefore consistent, and the system can thus be solved in the field of coefficients $\mathbb{C}(X_0)$. Specialize b_λ^i to such solutions, so that $(b_j^i) \in M_{n \times n}(\mathbb{C}(X_0))$. When we apply v on $P_l(y_j^i) = f_l$ in K , we obtain the solutions a_λ^i to (2). Hence the $(a_j^i) - (b_j^i)$ is a solution to the homogeneous system associated to (2); but the left hand side of the equations in the system are the polynomials defining \mathfrak{g} , so $(a_j^i) - (b_j^i) \in \mathfrak{g}(k)$. \square \square

Remark 27. *We have*

$$\begin{array}{ccc} \mathbb{C}(y_j^i) & \xrightarrow{\dots\dots\dots} & K \\ \vdots & \nearrow & \downarrow \\ \mathbb{C}(y_j^i)^{Z(\tilde{G})} & & k \\ \downarrow & \nearrow & \\ \mathbb{C}(X_0) & & \end{array}$$

Definition 7. *We say that the projective Fano curve X_0 is degenerate if it is not 1-dimensional, i.e. if $\mathbb{C}(X_0) = \mathbb{C}$.*

Proposition 9. *The projective Fano curve is degenerate if and only if G is connected and $\mathbb{C}(y_j^i)$ corresponds to the field of rational functions over a coset of G in $GL_n(\mathbb{C})$.*

Proof. Let $I_{\mathbb{C}}$ be the kernel of the evaluation map $\Psi \vdash: \mathbb{C}[X_j^i, \frac{1}{\det}] \rightarrow K$ sending $X_j^i \mapsto y_j^i$. Then $I_{\mathbb{C}}$ is a prime ideal. $I_{\mathbb{C}}$ is invariant under the G -action, so passing to the quotient we see that $(I_{\mathbb{C}})^G = I_G$ is the kernel of the restriction of the evaluation map to $\mathbb{C}[X_j^i, \frac{1}{\det}]^G \rightarrow K^G = k$. Again, I_G is a prime ideal. Because the Fano curve is degenerate, the maximal homogeneous ideal J contained in I_G corresponds at the level of varieties to a line, and I_G to a point in that line. So we conclude that G acts transitively by left multiplication over the

subvariety of $\mathrm{GL}_n(\mathbb{C})$ defined by $I_{\mathbb{C}}$. In other words, $I_{\mathbb{C}}$ defines a coset of G . Finally, because $I_{\mathbb{C}}$ is prime, this coset is connected, whence G is connected. This proves the necessity in the statement of the proposition. The sufficiency follows immediately by noting that G acts transitively on its cosets. \square \square

Remark 28. *When G is connected the field of rational functions over a coset of G is isomorphic to $\mathbb{C}(G)$ because the coset and G are isomorphic varieties.*

Remark 29. *When $\mathbb{C}(X_0) = \mathbb{C}$, the system (2) is homogeneous so $(a_j^i) \in \mathfrak{g}(k)$.*

2.3 Projective Equivalence and Pullbacks

Let us consider the algebraic properties of projective equivalence and of pullbacks.

Let us start with projective equivalences. We put another meromorphic connection ∇_1 on $\Pi : E \rightarrow X$ and we assume that ∇ and ∇_1 are projectively equivalent. This means that there is a 1-dimensional meromorphic bundle $P : L \rightarrow X$ with connection ∇' such that ∇_1 can be identified with $\nabla' \otimes \nabla$. We make this identification explicit by fixing (as before) a global frame (e_1, \dots, e_n) for $\mathcal{E}(X)$ and a non-zero global section $s^1 \in \mathcal{L}^*(X)$ such that the mapping $V \mapsto s^1 \otimes V$ is a horizontal isomorphism $(E, \Pi, \nabla_1) \rightarrow (L \otimes_X E, P \otimes \Pi, \nabla' \otimes \nabla)$, where $s^1(s_1) = 1$.

Let $U \subseteq X$ be an open set avoiding the singularities of ∇ and of ∇' , and let $h \in \mathcal{L}(U)$ be such that $\nabla' h = 0$. We set $f := s^1(h)$. As usual, we choose (V_1, \dots, V_n) a local horizontal frame of $\mathcal{E}(U)$ with respect to ∇ , and $y_j^i = e^i(V_j)$. In particular the coordinates of $h \otimes V_j$ on the frame $(s_1 \otimes e_1, \dots, s_1 \otimes e_n)$ are $s^1 \otimes e^i(h \otimes V_j) = f y_j^i$.

Let $W_j \in \mathcal{E}(U)$ be defined for $j \in \{1, \dots, n\}$ by

$$W_j = f y_j^i e_i,$$

so that $s^1 \otimes e^i(s_1 \otimes W_j) = f y_j^i$ and $s_1 \otimes W_j = h \otimes V_j$. Thus

$$\nabla_1 W_j = \nabla' \otimes \nabla(s_1 \otimes W_j) = \nabla' \otimes \nabla(h \otimes V_j) = 0.$$

In particular, a Picard-Vessiot extension for ∇_1 is generated by $(f y_j^i)$.

Proposition 10. *Under the hypotheses of Compoint's theorem, if ∇ and ∇_1 are projectively equivalent then their projective Fano curves coincide.*

Proof. Let $P(X_j^i) \in \mathbb{C}[X_j^i]^G$ be homogeneous of degree d such that $P(y_j^i) = 0$. Then $P(f y_j^i) = f^d P(y_j^i) = 0$. \square \square

Now let us turn our attention to pullbacks. Let $\Pi_0 : E_0 \rightarrow X_0$ be an n -dimensional meromorphic vector bundle with connection ∇_0 given by

$$\nabla_{0v_0} e_i = -b_i^j e_j$$

for a fixed global frame (e_1, \dots, e_n) of $\mathcal{E}_0(X_0)$. The pullback to X of (E_0, Π_0, ∇_0) is (algebraically) defined by taking the tensor product

$$\mathcal{E}_0 \otimes_{\mathcal{M}_0} \mathcal{M}$$

and regarding it as a sheaf of differential \mathcal{M} -modules. In particular

$$\mathcal{E}_0 \otimes_{\mathcal{M}_0} \mathcal{M}(X) = \mathcal{E}_0(X_0) \otimes_{\mathbb{C}(X_0)} k.$$

Therefore if ∇ is the pullback of ∇_0 then we have:

$$\nabla_{\tilde{v}_0}(e_i \otimes 1) = -b_i^j(e_j \otimes 1)$$

where \tilde{v}_0 stands for the lifting of v_0 to $\mathcal{T}X$. Now if v is another derivation in $\mathcal{T}X$ then there is an $f_v \in k$ such that $v = f_v \tilde{v}_0$ and so $\nabla_v = f_v \nabla_{\tilde{v}_0}$. If we denote the basis $e_i \otimes 1$ by e_i , then we have:

$$\nabla_v e_i = -f_v b_i^j e_j$$

Remark 30. *If, as above, ∇ is the pullback of ∇_0 and (e_1, \dots, e_n) is a cyclic basis for ∇_{0v_0} , i.e. $\nabla_{0v_0} e_i = e_{i+1}$ for $i \in \{1, \dots, n-1\}$, then $(e_1 \otimes 1, \dots, e_n \otimes 1)$ is a cyclic basis for $\nabla_{\tilde{v}_0}$ but not for ∇_v . A cyclic basis for ∇_v is $(e_1 \otimes 1, f_v e_2 \otimes 1, \dots, f_v^{n-1} e_n \otimes 1)$. Now, if $(y_j^i e_i)_j$ is a full-system of solutions for ∇_0 then $(y_j^i e_i \otimes 1)_j$ is a full-system of solutions for ∇ , therefore a Picard-Vessiot extension for ∇ is generated by (y_j^i) too.*

2.4 Standard connections

Definition 8. *Let k_0 be a subfield of k . We say that ∇ is defined over k_0 if in (1)*

$$(a_j^i) \in M_{n \times n}(k_0) + \mathfrak{g}(k)$$

Remark 31. *It follows from the definition and Lemma 8 that, when ∇ is such that $\det(y_j^i) = 1$, the connection is defined over X_0 .*

Theorem 11. *If ∇ has reductive Galois group the connection is projectively equivalent to a connection defined over its projective Fano curve.*

Proof. We fix the notation as above; in particular the horizontal sections of ∇ satisfy the linear differential equation

$$v(f^i) = a_j^i f^j.$$

It is a well-known fact that if we tensor ∇ with the connection given by the equation

$$v(f) = -\frac{a_i^i}{n} f = -\frac{f}{n} \sum_i a_i^i$$

then the resulting connection has unimodular Galois group [16, Exercises 1.35.5]. In particular we may take (y_j^i) such that $\det(y_j^i) = 1$. Furthermore, [5, Proposition 2.2] guarantees that the Galois group remains reductive after tensoring. We may therefore assume that ∇ satisfies the hypotheses of Compoint's theorem. The result now follows from Lemma 8. \square \square

Remark 32. In view of Proposition 9, the classification of connections with degenerate Fano curve is very simple: they correspond to extensions isomorphic to $G(k)/k$. Indeed, they are just the pullbacks of extensions $\mathbb{C}(G)/\mathbb{C}$, where G is a connected algebraic group.

Definition 9. We say that ∇ is standard if $K = \mathbb{C}(y_j^i)$ and $k = \mathbb{C}(X_0)$.

Corollary 12. Assume ∇ has reductive Galois group and its projective Fano curve is not degenerate. The connection ∇ is projectively equivalent to the pullback by a rational map of a standard connection over X_0 if and only if in Lemma 8

$$(a_j^i) \in M_{n \times n}(\mathbb{C}(X_0)).$$

In particular, if G is finite then ∇ is projectively equivalent to the pullback of a standard connection over X_0 .

Proof. After tensoring like in the proof of the theorem we obtain the following diagram:

$$\begin{array}{ccc} & & K \\ & \nearrow & \downarrow k \\ \mathbb{C}(y_j^i) & & k \\ \downarrow & \nearrow & \\ \mathbb{C}(X_0) & & \end{array}$$

If ∇ is the pullback of a standard connection over X_0 then in Lemma 8, a_λ^i is a solution to the system (2) in $\mathbb{C}(X_0)$ and so $(a_j^i) - (b_j^i) = 0$. Conversely $(a_j^i) \in M_{n \times n}(\mathbb{C}(X_0))$ defines a connection in a rank n bundle over X_0 and since $K = k(y_j^i)$, we conclude that ∇ is the pullback through the map $\mathbb{C}(X_0) \rightarrow k$.
The algebraic case follows from the fact that if G is finite then $\mathfrak{g} = 0$. $\square \quad \square$

3 The classifying ruled surfaces.

The relevance of having a classification of a collection of mathematical objects by other mathematical objects is that we can organize the former if we have a classification of the latter. In the case of standard equations this becomes a key point because, whereas the collection of standard equations for order two is classified by coverings of the sphere by the sphere, the collection of standard equations for higher order is infinite (see [3]) and so far no structure has been given to it. Furthermore, if an algorithmic implementation of the results in this paper is expected, as there is for the order two case, it would require a systematic way of classifying standard equations.

For the remainder of the article we will assume that the projective Fano curve X_0 is non-degenerate. Recall that this curve is defined by the homogeneous elements in $\mathbb{C}[X_j^i]^G$ vanishing when we evaluate them at (y_j^i) . The aim of

this section is to explain how we can associate a ruled surface to each class of projectively equivalent standard connections over X_0 .

Definition 10. A ruled surface is a surface Σ (a two dimensional \mathbb{C} -variety), together with a surjective map $\pi : \Sigma \rightarrow X_0$, where X_0 is a curve (a one dimensional \mathbb{C} -variety), such that the fibers $\pi^{-1}y$ are isomorphic to $\mathbb{P}^1(\mathbb{C})$, for every $y \in X_0$.

Consider an irreducible standard connection ∇_0 over X_0 with unimodular reductive Galois group G and fundamental system of solutions (y_j^i) ; and we fix $P(X_j^i) \in \mathbb{C}[X_j^i]^G$ such that $P(y_j^i) = 1$. Let $P_1, \dots, P_r \in \mathbb{C}[X_j^i]^G$ be homogeneous generators of the subalgebra of G -invariants in $\mathbb{C}[X_j^i]$. We denote by I^G the kernel of the evaluation \mathbb{C} -morphism

$$\begin{aligned} \Psi \upharpoonright_{\mathbb{C}[X_j^i]^G} : \mathbb{C}[X_j^i]^G &\longrightarrow \mathbb{C}(X_0) \\ P_l(X_j^i) &\longmapsto P_l(y_j^i) \quad \forall l \in \{1, \dots, r\}, \end{aligned}$$

which includes the homogeneous ideal J that defines the Fano curve. Geometrically we have a curve $V(I^G)$ in $(\mathbb{C}^{n \times n})^G$ generating the cone $V(J)$.

We embed $\mathbb{C}^{n \times n}$ into $\mathbb{P}^{n \times n}(\mathbb{C})$ by introducing the homogeneous coordinates $(z : x_j^i)$ for $\mathbb{P}^{n \times n}(\mathbb{C})$ and identifying $\mathbb{C}^{n \times n}$ with $z = 1$. In particular we put $X_j^i = \frac{x_j^i}{z}$. We extend the action of G on $\mathbb{C}^{n \times n}$ to $\mathbb{P}^{n \times n}(\mathbb{C})$ by declaring

$$(g_j^i) : z \mapsto z, \quad x_j^i \mapsto x_l^i g_j^l.$$

Now consider $\mathbb{P}^{n \times n}(\mathbb{C}) \times \mathbb{P}^{n \times n-1}(\mathbb{C})$, where the second factor has homogeneous coordinates (w_j^i) . Again, we extend the action of G by declaring that on the second factor we have

$$(g_j^i) : w_j^i \mapsto w_l^i g_j^l.$$

So that the variety Y defined by the homogeneous equations $x_j^i w_\kappa^\iota - x_\kappa^\iota w_j^i$ is the blown-up of $\mathbb{P}^{n \times n}(\mathbb{C})$ at 0 and it is invariant under the G -action. Indeed,

$$(g_j^i)(x_j^i w_\kappa^\iota - x_\kappa^\iota w_j^i) = g_j^l g_\kappa^\lambda (x_l^i w_\lambda^\iota - x_\lambda^\iota w_l^i).$$

So we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & \mathbb{P}^{n \times n}(\mathbb{C}) \times \mathbb{P}^{n \times n-1}(\mathbb{C}) \\ & \searrow & \downarrow \\ & & \mathbb{P}^{n \times n}(\mathbb{C}) \end{array}$$

of G -morphisms which yields

$$\begin{array}{ccc} Y^G & \xrightarrow{\quad} & (\mathbb{P}^{n \times n}(\mathbb{C}) \times \mathbb{P}^{n \times n-1}(\mathbb{C}))^G \\ & \searrow \varpi & \downarrow \\ & & (\mathbb{P}^{n \times n}(\mathbb{C}))^G \end{array}$$

We set $\Sigma := \varpi^{-1} \overline{V(J)}$.

Lemma 13. *The map:*

$$\begin{aligned}\pi : \Sigma &\longrightarrow X_0 \\ (z : x_j^i, w_\kappa^t) \cdot G &\longmapsto (w_\kappa^t) \cdot G\end{aligned}$$

defines a ruled surface.

Proof. Let $(q_\kappa^t) \cdot G \in X_0$. Let $R(X_j^i) \in \mathbb{C}[X_j^i]^G$ be homogeneous and such that $R(q_\kappa^t) \neq 0$. If $Q(X_j^i) \in \mathbb{C}[X_j^i]^G$ is such that $Q(q_\kappa^t) = 0$, and if $(p_j^i) \cdot G \in (\mathbb{C}^{n \times n})^G$ is such that $(z : p_j^i, q_\kappa^t) \in \Sigma$, then

$$Q(p_j^i)R(q_\kappa^t) - R(p_j^i)Q(q_\kappa^t) = 0,$$

implying that $Q(p_j^i) = 0$. Therefore $(p_j^i) \cdot G = (q_\kappa^t) \cdot G$. We have then $\varpi^{-1}(q_\kappa^t) = \overline{\{(z : q_j^i, q_\kappa^t) \cdot G \mid z \in \mathbb{C}\}}$, so that $\varpi^{-1}(q_\kappa^t)$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$. $\square \quad \square$

Remark 33. *It follows from the proof that the ruled surface Σ is determined by the homogeneous polynomials*

$$Q(x_j^i)R(w_\kappa^t) - R(x_j^i)Q(w_\kappa^t), \quad Q(X_j^i) \in J, \quad R(X_j^i) \in \mathbb{C}[X_j^i]^G.$$

In particular two projectively equivalent standard connections have the same ruled surface because their ideals J coincide. Now we will prove that the connection is uniquely determined then by $\pi : \Sigma \rightarrow X_0$ together with the curve $\varpi^{-1}(\overline{V(I_G)})$.

Lemma 14. *To a standard connection ∇_0 over X_0 with unimodular reductive Galois group we associate a ruled surface $\pi : \Sigma \rightarrow X_0$. If two standard connections are projectively equivalent then their associated ruled surfaces coincide. Among the connections associated to $\pi : \Sigma \rightarrow X_0$, ∇_0 is characterized by the curve $\varpi^{-1}(\overline{V(I_G)})$.*

Proof. The first two statements in the lemma follows from the previous one and the remark just above. Now if we restrict ourselves to the open set $U = \{z \neq 0\}$ on $\mathbb{P}^{n \times n}(\mathbb{C})$ and to the open set $U_0 = \{P(q_\kappa^t) \neq 0\}$ on X_0 , then the map (using the notation from Compoint's Theorem, Theorem 5)

$$\begin{aligned}\mathbb{C}[X_j^i]^G &\longrightarrow \mathbb{C}(X_0) \\ P_i &\longmapsto f_i\end{aligned}$$

sends, on the level of varieties, U_0 into $V(I_G)$. This map is the one defining the maximal differential ideal I in Compoint's Theorem. $\square \quad \square$

Remark 34. *The ruled surfaces obtained by blowing-up the vertex of a cone can also be obtained by taking the projective bundle defined by a rank-two holomorphic vector bundle over the base curve X_0 . In order to obtain Σ through a vector bundle, we are going to construct a two-dimensional vector bundle over $\mathbb{P}^{n \times n-1}(\mathbb{C})^G$ which we will then pullback to X_0 . We adapt the exposition in [12,*

Example V.2.11.4] to our specific setting.

Given a positive integer N we denote by $\mathbb{C}[X_j^i]_{\geq N}^G$ the $\mathbb{C}[X_j^i]^G$ -algebra of polynomials of degree greater than or equal to N . Note that

$$\mathbb{C}[X_j^i]^G = \bigoplus_{m \in \mathbb{N}} \mathbb{C}[X_j^i]_m^G$$

where $\mathbb{C}[X_j^i]_m^G$ denotes the homogeneous polynomial of degree m and

$$\mathbb{C}[X_j^i]_{[N]}^G = \bigoplus_{m \in \mathbb{N}} \mathbb{C}[X_j^i]_{mN}^G$$

can both be used as homogeneous coordinates of $\mathbb{P}^{n \times n - 1}(\mathbb{C})^G$.

Now assume $P_1, \dots, P_r \in \mathbb{C}[X_j^i]^G$ is a minimal set of homogeneous generators of $\mathbb{C}[X_j^i]^G$. Denote by n_i the degree of P_i , $i \in \{1, \dots, r\}$, and by $N = [n_1, \dots, n_r]$ the least common multiple of these degrees. Set $N = N_i n_i$

$$M_0 = \mathbb{C}[X_j^i]^G \cdot Z \simeq \mathbb{C}[X_j^i]^G$$

and

$$M_N = \sum_{m_1 n_1 + \dots + m_r n_r = N} \mathbb{C}[X_j^i]^G \prod_l P_l^{m_l} \simeq \mathbb{C}[X_j^i]_{\geq N}^G.$$

Each of these two free graded $\mathbb{C}[X_j^i]^G$ -modules define a rank-one holomorphic vector bundles $L_0 \rightarrow \mathbb{P}^{n \times n - 1}(\mathbb{C})^G$ and $L_N \rightarrow \mathbb{P}^{n \times n - 1}(\mathbb{C})^G$ respectively. Let $V \rightarrow \mathbb{P}^{n \times n - 1}(\mathbb{C})^G$ be the fibred sum of L_0 and L_N . Denote by \mathcal{O} the sheaf of holomorphic sections of $L_0 \rightarrow \mathbb{P}^{n \times n - 1}(\mathbb{C})^G$ and by $\mathcal{O}(N)$ the sheaf of holomorphic sections of $L_N \rightarrow \mathbb{P}^{n \times n - 1}(\mathbb{C})^G$. In particular we have that the sheaf of sections of $V \rightarrow \mathbb{P}^{n \times n - 1}(\mathbb{C})^G$ is $\mathcal{V} = \mathcal{O} \oplus \mathcal{O}(N)$.

We want to construct the graded $\mathbb{C}[X_j^i]^G$ -algebra given by the symmetric algebra of V (cf. [12, Section II.7]). Set

$$\mathbb{C}[X_j^i]^G[Z] \otimes_{\mathbb{C}[X_j^i]^G} \mathbb{C}[X_j^i]^G \left[\prod_l P_l^{m_l} \mid m_1 n_1 + \dots + m_r n_r = N \right] =: \mathbb{C}[X_j^i]^G[Z][Y_j^i]_{\geq N}^G$$

where Z will have degree 1 and $\mathbb{C}[Y_j^i]_{\geq N}^G$ will be graded by $\deg - (N - 1)$. To obtain the desired algebra we take the quotient algebra defined by the $\mathbb{C}[X_j^i]^G$ -morphism

$$\begin{aligned} \mathbb{C}[X_j^i]^G[Z][Y_j^i]_{\geq N}^G &\longrightarrow \mathbb{C}[X_j^i]^G \\ Z &\longmapsto 1 \\ Q(Y_j^i) &\longmapsto Q(X_j^i), \quad \text{for } Q(Y_j^i) \in \mathbb{C}[Y_j^i]_{\geq N}^G. \end{aligned}$$

The biggest homogeneous ideal in the kernel of this morphism is generated by the elements

$$Q(X_j^i)R(Y_j^i) - R(X_j^i)Q(Y_j^i) \quad Q(X_j^i), R(X_j^i) \in \mathbb{C}[X_j^i]_{\geq N}^G,$$

which would yield the same ideal defining Y^G if instead of using $\mathbb{C}[X_j^i]^G$ as homogeneous coordinate ring of $\mathbb{P}^{n \times n - 1}(\mathbb{C})^G$ we use $\mathbb{C}[X_j^i]_{[N]}^G$. So we conclude that Y^G corresponds to the projective bundle defined by $V \rightarrow \mathbb{P}^{n \times n - 1}$.

Remark 35. Let us denote the pullback to X_0 of \mathcal{V} (resp. of \mathcal{O} , of $\mathcal{O}(N)$) by \mathcal{V}_{X_0} (resp. by \mathcal{O}_{X_0} , by $\mathcal{O}_{X_0}(N)$). Then as Y^G is obtained as the projective bundle from \mathcal{V} , the portion Σ over X_0 is obtained by taking the projective bundle defined by \mathcal{V}_{X_0} . In particular, since $\mathcal{V}_{X_0} = \mathcal{O}_{X_0} \oplus \mathcal{O}_{X_0}(N)$ and \mathcal{O}_{X_0} is the sheaf of holomorphic functions over X_0 , we conclude that Σ is uniquely determined by $\mathcal{O}_{X_0}(N)$.

It is customary to normalize \mathcal{V} by tensoring with $\mathcal{O}(-N) = \text{Hom}_{\mathcal{O}}(\mathcal{O}(N), \mathcal{O})$. This does not change the induced projective bundle, so that the rank two vector bundle inducing Σ is $\mathcal{O}_{X_0} \oplus \mathcal{O}_{X_0}(-N)$.

Theorem 15. Let ∇_0 and ∇ be two standard connections, defined on the same meromorphic vector bundle $E \rightarrow X_0$, with reductive Galois groups such that their projective Galois groups are isomorphic. Then ∇_0 and ∇ are projectively equivalent if and only if their associated ruled surface coincide.

Proof. The necessity has been already established in Lemma 14. We retain the same notation as above for the connection ∇_0 . In particular $\pi : \Sigma \rightarrow X_0$ is the associated ruled surface, and $P(y_j^i) = 1$ for some fixed $P(X_j^i) \in \mathbb{C}[X_j^i]^G$. Because $\overline{V(J)}$ is a projective variety of dimension two, we have that the irreducible variety $\overline{V(I_G)}$ is in the closure of the curve $V(J) \cap V(P(X_j^i) - 1)$, and they coincide if $V(J) \cap V(P(X_j^i) - 1)$ is irreducible. Without loss of generality we may assume that ∇_0 and ∇ are normalized so that their n -th exterior product has rational sections. In such a case $P(X_j^i) = \det(X_j^i)$, $P(X_j^i) - 1$ is prime and $\overline{V(I_G)} = \overline{V(J) \cap V(P(X_j^i) - 1)}$. It follows that the curve in Σ defining ∇_0 and ∇ coincide (cf. Lemma 14), whence $\nabla = \nabla_0$. \square \square

Computing the ruled surfaces

Although the proof above helps us characterizing the ruled surfaces arising from standard equations, the proof offers little towards effectively computing these. In fact, even if we manage to get the N in the expression $\mathcal{V} = \mathcal{O} \oplus \mathcal{O}(N)$, it is not easy to obtain the pullback and the blow up to explicitly get $\mathcal{V}_{X_0} = \mathcal{O}_{X_0} \oplus \mathcal{O}_{X_0}(N)$. To carry out the computations we will rely on the Nash blowing-up (cf. [15]).

Definition 11. Let X be a subvariety of dimension r of a complex variety of dimension m . Let S be the set of singular points of X and X_S its complement in X . Set

$$\begin{aligned} \eta : X_S &\longrightarrow X \times G_r^m \\ x &\longmapsto (x, T_{X,x}) \end{aligned}$$

where G_r^m is the Grassmanian of r -planes in m -space and $T_{X,x}$ is the tangent to X at x seen as an r -plane inside the tangent to the ambient variety at x . The Nash blow-up is

$$\varpi : X^* \longrightarrow X$$

where X^* is the closure of $\eta(X_S)$ and ϖ is given by the first projection.

Remark 36. *Nash blowing-up is a monoidal transformations [15, Theorem 1], therefore it transforms the variety X into a bi-rationally equivalent space [12, Proposition 7.16]. The same is true for the blow-up at a point. Thus, the spaces obtained from the cone defined by (y_j^i) via Nash blowing-up or one point blow-up are birationally equivalent.*

Let $P(X_j^i)$ be a homogeneous function of degree d such that $P(y_j^i) = 0$, then

$$v(P(y_j^i)) = v(y_\kappa^\iota) \frac{\partial P}{\partial X_\kappa^\iota}(y_j^i) = 0;$$

and, from Euler's Theorem

$$dP(y_j^i) = y_\kappa^\iota \frac{\partial P}{\partial X_\kappa^\iota}(y_j^i) = 0.$$

Therefore at the regular points of the cone defined by (y_j^i) , the tangent space is spanned by (y_j^i) and $(v(y_j^i))$. From where we obtain:

Proposition 16. *Let ∇ be a standard connection with reductive and unimodular Galois group G , and let P_0, \dots, P_r be a set of homogeneous generators of $\mathbb{C}[X_j^i]^G$. If (y_j^i) is a fundamental system of solutions, then the ruled surface associated to ∇ corresponds to the projective bundle defined by the rank-two vector bundle $\mathcal{L} \oplus \mathcal{L}'$ where \mathcal{L} is defined by the map (cf.[12, Proposition 7.1])*

$$\begin{aligned} \mathbb{C}[X_j^i]^G &\longrightarrow k \\ P_i(X_j^i) &\longmapsto P_i(y_j^i) \end{aligned}$$

and \mathcal{L}' is defined by

$$\begin{aligned} \mathbb{C}[X_j^i]^G &\longrightarrow k \\ P_i(X_j^i) &\longmapsto P_i(v(y_j^i)) \end{aligned}$$

Proof. Identifying all the tangent spaces to $\mathbb{C}^{n \times n}$ with the tangent at the origin, G acts on the tangent spaces in the same way as it acts on $\mathbb{C}^{n \times n}$. This shows the map η in the definition of the Nash blowing-up as a G -morphism. Taking quotients by the G -action, the 2-space spanned by (y_j^i) and $(v(y_j^i))$ maps to the space spanned by $(y_j^i) \cdot G$ and $(v(y_j^i)) \cdot G$. \square

Appendix: The algebraic case

Computing the genus of the projective curve defined by (y_j^i) .

Remark 37. *The content of this part of the appendix is a generalization of the presentation of [1, Lemma 1.5] to higher order.*

Given a n -th order irreducible linear differential equation $L(y) = 0$ over X with algebraic solutions, and a point $p \in X$, we denote by

$$E(L, p) = \{\alpha_{1,p}, \alpha_{2,p}, \dots, \alpha_{n,p}\}$$

the collection of generalized exponents of $L(y) = 0$ at p (i.e. the roots of the indicial polynomial, which are all rational because the solutions are algebraic) ordered so that $\alpha_{i,p} < \alpha_{j,p}$ if $i < j$. We set

$$\Delta(L, p) = \alpha_{n,p} - \alpha_{1,p} - (n - 1)$$

and we let $e(L, p)$ be the least common denominator of $E(L, p) - E(L, p) = \{a - b \mid a, b \in E(L, p)\}$ (i.e. the smallest $m \in \mathbb{N}$ such that $m(E(L, p) - E(L, p)) \subseteq \mathbb{Z}$). If $S \subseteq X$ we set

$$\Delta(L, S) := \sum_{p \in S} \Delta(L, p).$$

As $E(L, p) = \{0, 1, \dots, n - 1\}$ for almost all $p \in X$, we have $\Delta(L, p) = 0$ for all but finitely many $p \in X$, and for a sufficiently large S , the number $\Delta(L, S)$ attains limiting value $\Delta(L)$.

Lemma 17. *Let $f : X \rightarrow X_0$ be a morphism of compact Riemann surfaces of degree M , and assume that $L(y) = 0$ is the pullback of $L_0(y) = 0$ through f . Furthermore, assume all solutions to $L_0(y) = 0$ are algebraic. Then, if g (resp. g_0) denotes the genus of X (resp. of X_0), we have:*

$$M\left(\frac{\Delta(L_0)}{n-1} - 2(g_0 - 1)\right) = \frac{\Delta(L)}{n-1} - 2(g - 1).$$

Proof. Let $S_0 \in X_0$ be a finite collection of points containing all ramifications of f and all singularities of $L_0(y) = 0$, and set $S := f^{-1}(S_0)$. So, if e_{p_0} denotes the ramification index of p_0 in f , then

$$\begin{aligned} \Delta(L, f^{-1}(p_0)) &= \sum_{p|p_0} \Delta(L, p) \\ &= \sum_{p|p_0} \alpha_{n,p} - \alpha_{1,p} - (n - 1) \\ &= \sum_{p|p_0} e_{p_0}(\alpha_{n,p_0} - \alpha_{1,p_0}) - (n - 1) \\ &= \frac{M}{e_{p_0}}(e_{p_0}(\alpha_{n,p_0} - \alpha_{1,p_0}) - (n - 1)) \\ &= M(\Delta(L_0, p_0) + (n - 1)) - \text{Card}(f^{-1}(p_0))(n - 1). \end{aligned}$$

Thus $\Delta(L, f^{-1}(p_0)) + (n - 1)\text{Card}(f^{-1}(p_0)) = M(\Delta(L_0, p_0) + (n - 1))$ and

$$\Delta(L, S) + (n - 1)\text{Card}(S) = M(\Delta(L_0, S_0) + (n - 1)\text{Card}(S_0))$$

As S_0 contains all ramifications of f , the Hurwitz genus formula implies

$$2(g-1) - 2M(g_0-1) = M \text{ Card}(S_0) - \text{Card}(S).$$

Combining the last two equalities we obtain the desired conclusion by noticing that S_0 contains all singularities of $L_0(y) = 0$. \square \square

Proposition 18. *Under the hypotheses of the lemma we have:*

$$\sum_{p_0 \in S_0} \left(\frac{1}{e(L_0, p_0)} - 1 \right) = 2(g_0 - 1) - \frac{2(g-1)}{M}.$$

Proof. Suppose f corresponds to the field extension

$$\mathbb{C}(X_0) \subseteq \mathbb{C}(X_0) \left[\frac{y_1}{y_n}, \dots, \frac{y_{n-1}}{y_n} \right],$$

where y_1, \dots, y_n denote a full system of solutions of $L_0(y) = 0$. Then $e_{p_0} = e(L_0, p_0)$, and it follows that

$$\begin{aligned} \frac{1}{n-1} (M\Delta(L_0) - \Delta(L)) &= \frac{1}{n-1} \left\{ M \sum_{p_0 \in S_0} (\alpha_{n,p_0} - \alpha_{1,p_0} - (n-1)) \right. \\ &\quad \left. - M \sum_{p_0 \in S_0} (\alpha_{n,p_0} - \alpha_{1,p_0} - \frac{n-1}{e(L_0, p_0)}) \right\} \\ &= \frac{M}{n-1} \sum_{p_0 \in S_0} \left(\frac{n-1}{e(L_0, p_0)} - (n-1) \right) \\ &= M \sum_{p_0 \in S_0} \left(\frac{1}{e(L_0, p_0)} - 1 \right). \end{aligned}$$

The statement now follows from the previous lemma. \square \square

Example 1. *Consider the three following equations: Ulmer's G_{54} equation [17]*

$$y''' + \frac{3(3x^2-1)}{x(x-1)(x+1)}y'' + \frac{221x^4-206x^2+5}{12x^2(x-1)^2(x+1)^2}y' + \frac{374x^6-673x^4+254x^2+5}{54x^3(x-1)^3(x+1)^3}y = 0$$

with singular points at 0, 1, -1 and ∞ , with respective exponents

$$\left\{ -\frac{1}{6}, \frac{1}{3}, \frac{4}{3} \right\}, \quad \left\{ -\frac{1}{6}, \frac{1}{3}, \frac{4}{3} \right\}, \quad \left\{ -\frac{1}{6}, \frac{1}{3}, \frac{4}{3} \right\}, \quad \left\{ -\frac{1}{6}, \frac{1}{3}, \frac{4}{3} \right\};$$

the Geiselmann-Ulmer $F_{36}^{SL_3}$ equation [8]

$$y''' + \frac{5(9x^2+14x+9)}{48x^2(x+1)^2}y' - \frac{5(81x^3+185x^2+229x+81)}{432x^3(x+1)^3}y = 0$$

with singular points at 0, 1 and ∞ , with respective exponents

$$\left\{ 1, \frac{3}{4}, \frac{5}{4} \right\}, \quad \left\{ \frac{5}{6}, \frac{11}{6}, \frac{1}{3} \right\}, \quad \left\{ -1, \frac{-3}{4}, \frac{-5}{4} \right\};$$

and the equation

$$y''' + \frac{1}{48} \frac{41z^2 - 50z + 45}{(z-1)^2 z^2} y' - \frac{1}{432} \frac{364z^3 - 665z^2 + 1030z - 405}{(z-1)^3 z^3} y = 0$$

with singularities at 0, 1 and ∞ with respective exponents

$$\left\{\frac{3}{4}, 1, \frac{5}{4}\right\}, \quad \left\{\frac{1}{2}, 1, \frac{3}{2}\right\}, \quad \left\{-\frac{4}{3}, -\frac{13}{12}, -\frac{7}{12}\right\}.$$

For the first equation we have $g_0 = 0$, $M = |PG_{54}| = 18$ and $e_0 = e_1 = e_{-1} = e_\infty = 2$, so that

$$\sum_{i \in \{0, 1, -1, \infty\}} \left(1 - \frac{1}{e_i}\right) = 2;$$

for the second equation we have $g_0 = 0$, $M = |F_{36}| = 36$ and $e_0 = e_\infty = 4$, $e_1 = 2$ so that

$$\sum_{i \in \{0, 1, \infty\}} \left(1 - \frac{1}{e_i}\right) = 2;$$

and, for the third equation we again have $g_0 = 0$, $M = |F_{36}| = 36$ and $e_0 = e_\infty = 4$, $e_1 = 2$. This tells us that the algebraic extension given by the ratio of solutions is a curve of genus 1. Indeed, the three equations are related to the generalized hypergeometric equation defining ${}_3F_2(-\frac{1}{12}, \frac{1}{6}, \frac{2}{3}, \frac{1}{2}, \frac{3}{4}; z)$, i.e.

$$y''' + \frac{3}{4} \frac{5z-3}{z(z-1)} y'' + \frac{1}{24} \frac{43z-9}{z^2(z-1)} y' - \frac{1}{108z^2(z-1)} y = 0.$$

The first of our equations is projectively equivalent to the pullback of this one by the map $z(x) = \frac{1}{16} \frac{(x^2+1)^4}{x^2(x+1)^2(x-1)^2}$; the second one is projectively equivalent to the pullback by the map $z(x) = \frac{4(x-1)}{x^2}$; and the third one is the normalized form which is standard. If one takes three linearly independent solutions to each of these equations, we can see that they satisfy a homogeneous equation with coefficients in \mathbb{C} of degree 3 in three variables defining an elliptic curve.

Example 2. Consider the following two equations. The first is van Hoeij's $H_{72}^{SL_3}$ equation [17], i.e.

$$\begin{aligned} 0 = & y''' + \frac{21x^2 - 24x - 1}{(3x^2 + 1)(x-1)} y'' + \frac{1}{48} \frac{4437x^3 - 5973x^2 + 171x - 683}{(3x^2 + 1)^2(x-1)} y' \\ & + \frac{1}{216} \frac{13338x^4 - 22647x^3 + 1983x^2 - 7297x - 737}{(3x^2 + 1)^3(x-1)} y. \end{aligned}$$

The singular points are 1 (which actually is an apparent singularity), $\frac{i\sqrt{3}}{3}$, $-\frac{i\sqrt{3}}{3}$ and ∞ , with respective exponents

$$\{0, 1, 3\}, \quad \left\{-\frac{7}{12}, -\frac{1}{3}, -\frac{1}{12}\right\}, \quad \left\{-\frac{7}{12}, -\frac{1}{3}, -\frac{1}{12}\right\}, \quad \left\{\frac{13}{12}, \frac{4}{3}, \frac{19}{12}\right\}.$$

The second is

$$y''' + \frac{1}{432} \frac{405z^2 - 469z + 384}{(z-1)^2 z^2} - \frac{1}{11664} \frac{10935z^3 - 18803z^2 + 27196z - 10368}{(z-1)^3 z^3} = 0.$$

Here the singular points are 0, 1 and ∞ , with respective exponents

$$\left\{\frac{2}{3}, 1, \frac{4}{3}\right\}, \quad \left\{\frac{5}{9}, \frac{8}{9}, \frac{14}{9}\right\}, \quad \left\{-\frac{5}{4}, -1, -\frac{3}{4}\right\}.$$

For the first equation we have $g_0 = 0$, $M = |H_{72}| = 72$ and $e_{\frac{i\sqrt{3}}{3}} = e_{-\frac{i\sqrt{3}}{3}} = e_\infty = 4$, so that

$$\sum_{j \in \{\frac{i\sqrt{3}}{3}, -\frac{i\sqrt{3}}{3}, \infty\}} \left(1 - \frac{1}{e_j}\right) = \frac{9}{4} = \frac{2(10-1)}{M} + 2.$$

For the second equation we have $g_0 = 0$, $M = |H_{216}| = 216$ and $e_0 = e_1 = 3$, $e_\infty = 4$, so that

$$\sum_{i \in \{0, 1, \infty\}} \left(1 - \frac{1}{e_i}\right) = \frac{25}{12} = \frac{2(10-1)}{M} + 2.$$

This tells us that the algebraic extension given by the ratio of solutions is a curve of genus 10. Indeed, the two equations are related to the generalized hypergeometric equation ${}_3F_2(-\frac{1}{36}, \frac{2}{9}, \frac{17}{36}, \frac{1}{3}, \frac{2}{3}; z)$, i.e.

$$y''' + \frac{1}{3} \frac{11z-6}{z(z-1)} y'' + \frac{1}{432} \frac{-96+757z}{z^2(z-1)} y' - \frac{17}{5832} \frac{1}{z^2(z-1)} = 0.$$

The first of our equations is projectively equivalent to the pullback of this one by the map $z(x) = \frac{1}{2} \frac{(x+1)^3}{(1+3x^2)}$; whereas the second one is the normalized form and is standard. If one takes three linearly independent solutions to each of these equations, we can see that they satisfy a homogeneous equation of degree 6 in three variables defining a curve of genus 10.

The ruled surfaces for the second order standard equations

The standard equation St_G of second order for the Galois group G is [3]

$$y'' + \left(\frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{x(x-1)} \right) y = 0$$

where $a = \frac{1-\lambda^2}{4}$, $b = \frac{1-\mu^2}{4}$, and $c = \frac{\lambda^2+\mu^2-1-\nu^2}{4}$; and, (λ, μ, ν) is $(1/3, 1/2, 1/3)$ if $G = A_4$, $(1/3, 1/2, 1/4)$ if $G = S_4$, $(1/3, 1/2, 1/5)$ if $G = A_5$, and $(1/2, 1/2, 1/n)$ if $G = D_{2n}$. Now, if (y_1, y_2) are the solutions to the equation

$$y'' - fy = 0,$$

then (y'_1, y'_2) are the solutions to the equation

$$y'' - \frac{f'}{f}y' - fy = 0.$$

In this way we can easily compute \mathcal{L} and \mathcal{L}' from Proposition 16 using the algorithm in [13].

Example 3. For St_{A_4} the invariants of degree 24 are spanned by

$$x^8(x-1)^8, \quad x^8(x-1)^7, \quad x^9(x-1)^6, \quad \text{and } x^8(x-1)^6$$

so \mathcal{L} is generated by $1, x, x-1, (x-1)^2$, hence $\mathcal{L} = \mathcal{O}(2)$. On the other hand the equation St'_{A_4} with solutions the derivative of the solutions to St_{A_4} has invariants of degree 24 spanned by

$$\frac{f_6^4}{(x-1)^{16}x^{16}}, \quad \frac{f_6^2 f_{12}}{(x-1)^{17}x^{16}}, \quad \frac{f_8^3}{(x-1)^{18}x^{15}}, \quad \text{and } \frac{f_{12}^2}{(x-1)^{18}x^{16}}$$

where $f_l, l \in \{6, 8, 12\}$, is a polynomial of degree l ; so \mathcal{L}' is generated by $1, x \frac{f_8^3}{f_{12}^2}, (x-1) \frac{f_6^2}{f_{12}}, (x-1)^2 \frac{f_6^4}{f_{12}^2}$, hence $\mathcal{L}' = \mathcal{O}(26)$. The ruled surface corresponding to A_4 is the projective bundle defined by $\mathcal{O}(2) \oplus \mathcal{O}(26)$, which is the same as the one defined by $\mathcal{O} \oplus \mathcal{O}(24)$.

Similar computations shows that:

- for A_4 the ruled surface is $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(26))$;
- for S_4 the ruled surface is $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(25))$;
- for A_5 the ruled surface is $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(61))$;
- for D_{2n} the ruled surface is $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(2[2n+1]))$, if $2 \nmid n$; and,
- for D_{2n} the ruled surface is $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(2n+1))$, if $2 \mid n$.

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